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New examples of Willmore tori in S^4

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Abstract
Using the (generalized) Darboux transformation in the case of the Clifford torus, we construct for all Pythagorean triples $(p, q, n) \in \mathbb{Z}^3$ a $\mathbb{CP}^3$-family of Willmore tori in S^4 with Willmore energy $2(n\pi)^2$.

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(Some figures in this article are in colour only in the electronic version)

1. Introduction

Classical geometers such as Bianchi, Darboux and Bäcklund used local transformations to obtain new examples of a particular class of surfaces out of simple known ones by geometric constructions. For instance, the Darboux transformation was classically [6] defined for isothermic surfaces, that is surfaces which allow a conformal curvature line parametrization: two conformal immersions f and $f^\sharp$ form a classical Darboux pair if there exists a sphere congruence which envelopes both surfaces f and $f^\sharp$. In this case, both f and $f^\sharp$ are isothermic.

Relaxing the enveloping condition [3] one obtains a (generalized) Darboux transformation for conformal immersions $f : M \to S^4$ of a Riemann surface into the 4-sphere. Darboux transforms of a conformal immersion are obtained by prolongations of holomorphic sections in an associated quaternionic holomorphic line bundle. In the case when $f : T^2 \to S^4$ is a conformal torus with trivial normal bundle, the set of multipliers of holomorphic sections gives rise to a Riemann surface, the (multiplier) spectral curve of f. In particular, each point on the spectral curve gives a holomorphic section with the multiplier, and thus there exists at least a Riemann surface worthy of closed Darboux transforms $\hat{f} : T^2 \to S^2$ from the 2-torus into the 4-sphere.

In the case when the conformal immersion is given by a harmonicity condition, e.g. for constant mean curvature surfaces, Hamiltonian stationary Lagrangians or (constrained)

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Willmore surfaces, one obtains an associated family of flat connections d^μ for $\mu \in \mathbb{C}_*$. Parallel sections of d^μ give holomorphic sections in the associated quaternionic holomorphic line bundle, and thus give rise to special Darboux transforms—so-called μ-Darboux transforms [2 ,5, 10, 11]. In the case of the Clifford torus $f : M \to S^3$ parallel sections of the associated family of flat connections can be computed explicitly, and in this paper we obtain new Willmore tori in S^4 by constructing μ-Darboux transforms of a covering of the Clifford torus f.

2. The Darboux transformation

We briefly recall the Darboux transformation on a conformal immersion $f : M \to S^4$ of a Riemann surface into the 4-sphere [3]. To this end, we consider the 4-sphere $S^4 = \mathbb{HP}^1$ as the quaternionic projective line and identify $f : M \to S^4$ with the pullback $L = f^*\mathcal{T}$ of the tautological line bundle over $\mathbb{HP}^1$ by f, that is $L_p = f(p)$. The derivative of f can be identified with the map $\delta = \pi d|_L$ where $\pi : V \to V/L$ is the canonical projection of the trivial $\mathbb{H}^2$ bundle V, and d is the trivial connection on V. Moreover, f is a conformal immersion if and only if there exists a complex structure, $S \in \Gamma(\mathrm{End}(V))$, $S^2 = -1$, stabilizing L such that

$$*\delta = S\delta = \delta S\,, \tag{2.1}$$

where $*$ denotes the negative Hodge star operator. Complex structures S on V can be, and will be in the following, identified with sphere congruences [4, proposition 2]. The conformality condition (2.1) means geometrically that the sphere congruence S envelopes f, that is, S passes through f, and the tangent planes of f and S coincide at corresponding points in an oriented way. In particular, two immersions $f, f^\sharp : M \to S^4$ are classical Darboux transforms of each other, if there exists a complex structure $S \in \Gamma(\mathrm{End}(V))$ with $*\delta = S\delta = \delta S$ and $*\delta^\sharp = S\delta^\sharp = \delta^\sharp S$, where δ and $\delta^\sharp$ denote the derivatives of f and $f^\sharp$, respectively.

To obtain the Darboux transformation for conformal immersions $f : M \to S^4$, one relaxes the enveloping condition.

Definition 2.1 [3]. *Let $f : M \to S^4$ be a conformal immersion. Then a conformal map $\hat{f} : M \to S^4$ is called a Darboux transform of f if $f(p) \neq \hat{f}(p)$ for all $p \in M$ and if there exists a sphere congruence enveloping f and left enveloping $\hat{f}$, that is if there exists a complex structure $S \in \Gamma(\mathrm{End}(V))$, $S^2 = -1$, with*

$$*\delta = S\delta = \delta S \quad \textit{and} \quad *\hat{\delta} = S\hat{\delta}.$$

We shortly recall the construction of Darboux transforms: since f is a conformal immersion, that is in particular $*\delta = S\delta$, the complex structure S induces a complex structure $J = S_{V/L} \in \Gamma(\mathrm{End}(V/L))$, $J^2 = -1$, on the line bundle V/L.

Lemma 2.2 [3]. *Let $f : M \to S^4$ be a conformal immersion and J be the associated complex structure on V/L. Then*

$$D\varphi := (\pi \mathrm{d}\hat{\varphi})''$$

defines a (quaternionic) holomorphic structure on V/L. Here $\hat{\varphi}$ is an arbitrary lift of $\varphi = \pi\hat{\varphi} \in \Gamma(V/L)$, and

$$\omega'' = \tfrac{1}{2}(\omega + J * \omega)$$

denotes the $(0, 1)$ part of a 1-form $\omega \in \Omega^1(V/L)$ with respect to the complex structure J.

Indeed, D is well defined since f is conformal and thus $(\pi\, \mathrm{d}\psi)'' = \delta\psi'' = 0$ for $\psi \in \Gamma(L)$. We denote the set of holomorphic sections by $\ker D = H^0(V/L)$, and consider holomorphic sections of the pullback $\widetilde{V/L}$ of V/L to the universal cover $\tilde{M}$ of M.

Lemma 2.3 (see [3]). *Every holomorphic section, $\varphi \in H^0(\widetilde{V/L})$, of the canonical holomorphic bundle of a conformal immersion $f : M \to S^4$ has a unique lift $\hat{\varphi} \in \Gamma(\tilde{V})$ such that*

$$\pi\, \mathrm{d}\hat{\varphi} = 0\,, \tag{2.2}$$

where $\pi : V \to V/L$ is the canonical projection. This unique lift $\hat{\varphi}$ is called the prolongation of φ.

Moreover, if φ is nowhere vanishing, then $\hat{f} = \hat{\varphi}\mathbb{H} : \tilde{M} \to S^4$ is a Darboux transform of f.

To obtain closed Darboux transforms of f, we have to consider holomorphic sections with multiplier, that is, $\varphi \in \ker D \subset \Gamma(\widetilde{V/L})$ with

$$\gamma^*\varphi = \varphi h_\gamma, \qquad h_\gamma \in \mathbb{C}_*, \qquad \gamma \in \pi_1(M).$$

Note that the prolongation $\hat{\varphi}$ of φ has the same multiplier as φ so that, if φ has no zeros, $\hat{f} = \hat{\varphi}\mathbb{H} : M \to S^4$ defines, indeed, a smooth map from the Riemann surface M into the 4-sphere. If the holomorphic section φ has zeros, the zeros are isolated [8], and the line bundle $\hat{\varphi}\mathbb{H}$ extends continuously into the zero locus of φ. In this case, $\hat{f} = \hat{\varphi}\mathbb{H}$ is called a *singular* Darboux transform. In fact, all closed Darboux transforms of a conformal immersion are obtained this way.

Lemma 2.4 [3]. *A map $\hat{f} : M \to S^4$ is a (singular) Darboux transform of f if and only if $\hat{f}$ is obtained by the non-constant prolongation of a holomorphic section $\varphi \in H^0(\widetilde{V/L})$ with a multiplier.*

3. μ-Darboux transforms of Willmore surfaces

The *conformal Gauss map* of a conformal immersion $f : M \to S^4$ is a sphere congruence which envelopes f and has the same mean curvature vector $\mathcal{H}$ as f. In terms of the corresponding complex structure S, this reads as [4, theorem 2]

$$*\delta = S\delta = \delta S \quad \text{and} \quad \operatorname{im} A \subset \Omega^1(L)\,, \tag{3.1}$$

where the *Hopf fields* A and Q are defined by the decomposition of the derivative of S,

$$\mathrm{d}S = 2(*Q - *A),$$

into (1, 0) and (0, 1) parts:

$$(\mathrm{d}S)' = \tfrac{1}{2}(\mathrm{d}S - S * \mathrm{d}S) = -2 * A$$

and

$$(\mathrm{d}S)'' = \tfrac{1}{2}(\mathrm{d}S + S * \mathrm{d}S) = 2 * Q,$$

respectively. Since $S^2 = -1$, the Hopf fields satisfy

$$*A = SA = -AS \quad \text{and} \quad *Q = -SQ = QS. \tag{3.2}$$

Let now $f : M \to S^4$ be a Willmore surface, i.e., f is an immersion which is a critical point of the Willmore energy, $W(f) = \int_M |\mathcal{H}|^2\, \mathrm{d}A$, under variations with compact support. It is a well-known fact [7, 12] that f is Willmore if and only if the conformal Gauss map of f is harmonic. This can be expressed [4, proposition 5] by the condition

$$d * A = 0 \qquad \text{or, equivalently,} \qquad d * Q = 0.$$

Lemma 3.1 [8, lemma 6.3]**.** *Let $f : M \to S^4$ be a conformal immersion with the conformal Gauss map S and Hopf field A. Then f is Willmore if and only if the family of complex connections*

$$d^\mu = d + *A(S(a-1) + b) \tag{3.3}$$

is flat for all $\mu \in \mathbb{C}_*$. Here $\mathbb{C} = \mathrm{Span}\{1, I\}$, where I is the complex structure on V given by right multiplication by the imaginary quaternion i, and

$$a = \frac{\mu + \mu^{-1}}{2}, \qquad b = I\frac{\mu^{-1} - \mu}{2}.$$

Proof. Since d is the trivial connection and $[I, S] = 0$, the curvature of d^μ is given by

$$R^\mu = (d * A)(S(a-1) + b),$$

where we used that $Q \wedge A = 0$ by type considerations. Therefore, S is harmonic if and only if d^μ is flat. □

We consider now parallel sections of d^μ with multiplier that is $d^\mu\hat\varphi = 0$ and $\gamma^*\hat\varphi = \hat\varphi h_\gamma$, $h_\gamma \in \mathbb{C}_*$, $\gamma \in \pi_1(M)$. Denoting the projection of $\hat\varphi$ to V/L by $\varphi = \pi\hat\varphi \in \Gamma(V/L)$ and recalling (3.1) that $*A(S\hat\varphi(a-1) + \hat\varphi b) \in \Gamma(L)$, we obtain

$$\pi\, \mathrm{d}\hat\varphi = 0.$$

In particular, φ is a holomorphic section with multiplier, and $\hat\varphi$ is the prolongation of φ. Lemma 2.4 now shows that every d^μ-parallel section with multiplier gives rise to a (singular) Darboux transform of f. Note that $\hat L$ is smoothly defined since $\hat\varphi$ is nowhere vanishing, and the derivative of $\hat f$ is given by (3.3)

$$\hat\delta\hat\varphi = -\pi_{\hat L} * A(S(a-1) + b)\hat\varphi.$$

On the other hand, the holomorphic section, $\varphi = \pi\hat\varphi$, may have zeros: this happens exactly for $p \in M$ with $\hat L_p = L_p$. In this case, the derivative of $\hat f$ vanishes at p since A_p takes values in $L_p = \hat L_p$. In particular, every singular μ-Darboux transform $\hat f$ of f is branched.

Definition 3.2. *A (singular) Darboux transform $\hat f : M \to S^4$ which is given by a parallel section of d^μ is called a μ-Darboux transform of f.*

Although, in general, the Darboux transforms of a Willmore torus are not necessarily Willmore [1], immersed μ-Darboux transforms are [2].

4. The Clifford torus

In this paper, we shall compute all μ-Darboux transforms of the Clifford torus:

$$f : \mathbb{C}/\Gamma \to S^3, \qquad u + \mathrm{i}v \mapsto \tfrac{1}{\sqrt{2}}(\mathrm{e}^{\mathrm{i}u} + j\,\mathrm{e}^{\mathrm{i}v}),$$

where $\Gamma = 2\pi\mathbb{Z} + 2\pi\mathrm{i}\mathbb{Z}$ is the lattice in $\mathbb{C}$. Note that though f maps into the 3-sphere, the μ-Darboux transforms will be (branched) conformal immersions into the 4-sphere. Therefore, we will consider a map $f : M \to S^3$ into the 3-sphere with the inclusions

$$S^3 \hookrightarrow \mathbb{R}^4 = \mathbb{H} \qquad \text{and} \qquad \mathbb{H} \hookrightarrow \mathbb{HP}^1, \qquad x \mapsto \begin{pmatrix} x \\ 1 \end{pmatrix}$$

as a map into the 4-sphere. The associated line bundle of f is given by $L = \psi\mathbb{H}$ where

$$\psi = \begin{pmatrix} f \\ 1 \end{pmatrix}.$$

The derivative of L is given by

$$\delta\psi = \pi \begin{pmatrix} \mathrm{d}f \\ 0 \end{pmatrix},$$

so that f is conformal if and only if there exists *left* and *right normals* $N, R : M \to S^2$ with $*\mathrm{d}f = N\,\mathrm{d}f = -\mathrm{d}f\,R$. The mean curvature vector $\mathcal{H}$ of a conformal immersion f is given [4, section 7.2] by

$$\mathcal{H} = -N\bar{H},$$

where H is defined by $\mathrm{d}f\,H = (\mathrm{d}N)'$. Here $'$ denotes the $(1, 0)$ part with respect to the complex structure given by left multiplication by N, that is

$$\omega' = \tfrac{1}{2}(\omega - N * \omega).$$

In particular, the conformal Gauss map of f is given by $S = GS_0G^{-1}$ where

$$G = \begin{pmatrix} 1 & f \\ 0 & 1 \end{pmatrix} \quad \text{and} \quad S_0 = \begin{pmatrix} N & 0 \\ -H & -R \end{pmatrix}, \tag{4.1}$$

and the Hopf field $A = GA_0G^{-1}$ by

$$*A_0 = \frac{1}{4}\begin{pmatrix} 0 & 0 \\ \mathrm{d}H + H * \mathrm{d}f\,H + R * \mathrm{d}H - H * \mathrm{d}N & \mathrm{d}R + R * \mathrm{d}R \end{pmatrix}.$$

Let us now turn to the case when $f : \mathbb{C}/\Gamma \to S^3$ is the Clifford torus. Then f is a conformal immersion with left and right normals,

$$N(u, v) = j\,\mathrm{e}^{\mathrm{i}(v-u)} \qquad \text{and} \qquad R(u, v) = j\,\mathrm{e}^{\mathrm{i}(v+u)},$$

and mean curvature vector $\mathcal{H} = -N\bar{H}$ where

$$H = \frac{\sqrt{2}}{2}(\mathrm{e}^{-\mathrm{i}u} + j\,\mathrm{e}^{\mathrm{i}v}).$$

Moreover, f satisfies the following fundamental symmetries:

(i) $R = Hf, \quad N = fH$,

(ii) H is conformal with $*\mathrm{d}H = -R\,\mathrm{d}H = \mathrm{d}H\,N$.

Therefore, the Hopf field, $A = GA_0G^{-1}$, is given by

$$*A_0 = \frac{1}{4}\begin{pmatrix} 0 & 0 \\ \mathrm{d}H & 2\,\mathrm{d}Hf \end{pmatrix},$$

where we also used that $RH = HN$; see [4, section 7.2].

5. μ-Darboux transforms of the Clifford torus

To compute μ-Darboux transforms of the Clifford torus f we have to find parallel sections $\hat{\varphi} \in \Gamma(V)$ of the family of flat connections d^μ on the trivial $\mathbb{H}^2$ bundle V. We solve the differential equation, $d^\mu\hat{\varphi} = 0$, that is with (3.2) we solve

$$\mathrm{d}\hat{\varphi} = -A\hat{\varphi}(a - 1) - *A\hat{\varphi}b.$$

Putting $\phi := G^{-1}\hat{\varphi}$ we can equivalently find solutions of

$$d\phi = -A_0\phi(a-1) - *A_0\phi b - (dG)\phi\,, \tag{5.1}$$

where we used that $G^{-1}\,dG = dG$. Since the connections d^μ are complex, this leads to a system of complex differential equations: writing $\phi = \binom{\alpha}{\beta}$ and decomposing $\alpha = \alpha_1 + j\alpha_2,\ \beta = \beta_1 + j\beta_2,\ \alpha_1, \alpha_2, \beta_1, \beta_2 \in \Gamma(\underline{\mathbb{C}})$ with respect to the splitting $\mathbb{H} = \mathbb{C} + j\mathbb{C}$, we consider

$$\phi = \begin{pmatrix}\alpha_1\\ \alpha_2\\ \beta_1\\ \beta_2\end{pmatrix} \in \Gamma(\underline{\mathbb{C}}^4)$$

as a section of the trivial $\mathbb{C}^4$ bundle. After a lengthy but straightforward computation [9] we obtain the system of a linear partial differential equation with non-constant coefficients:

$$\phi_u = U\phi, \qquad \phi_v = V\phi, \tag{5.2}$$

where we denote by $()_u$ and $()_v$ the partial derivatives with respect to u and v respectively, and

$$U(u,v) = \frac{1}{4\sqrt{2}}\begin{pmatrix} 0 & 0 & -4\,\mathrm{i}e^{\mathrm{i}u} & 0\\ 0 & 0 & 0 & 4\,\mathrm{i}e^{-\mathrm{i}u}\\ \mathrm{i}e^{-\mathrm{i}u}b & \mathrm{i}e^{-\mathrm{i}v}(a-1) & \sqrt{2}\mathrm{i}(a-1+b) & \sqrt{2}\,\mathrm{i}e^{-\mathrm{i}(u+v)}(a-1-b)\\ \mathrm{i}e^{\mathrm{i}v}(a-1) & -\mathrm{i}e^{\mathrm{i}u}b & \sqrt{2}\,\mathrm{i}e^{\mathrm{i}(u+v)}(a-1-b) & -\sqrt{2}\mathrm{i}(a-1+b)\end{pmatrix}$$

$$V(u,v) = \frac{1}{4\sqrt{2}} \times \begin{pmatrix} 0 & 0 & 0 & -4\,\mathrm{i}e^{-\mathrm{i}v}\\ 0 & 0 & -4\,\mathrm{i}e^{\mathrm{i}v} & 0\\ \mathrm{i}e^{-\mathrm{i}u}(a-1) & -\mathrm{i}e^{-\mathrm{i}v}b & \sqrt{2}\mathrm{i}(a-1-b) & -\sqrt{2}\mathrm{i}\,e^{-\mathrm{i}(u+v)}(a-1+b)\\ -\mathrm{i}e^{\mathrm{i}v}b & -\mathrm{i}e^{\mathrm{i}u}(a-1) & -\sqrt{2}\,\mathrm{i}e^{\mathrm{i}(u+v)}(a-1+b) & -\sqrt{2}\mathrm{i}(a-1-b)\end{pmatrix}.$$

Lemma 5.1. *A section $\hat{\varphi} \in \Gamma(V)$ is parallel with respect to d^μ if and only if*

$$\eta := e^D G^{-1}\hat{\varphi}, \qquad D(u,v) := \mathrm{diag}(\mathrm{i}v, \mathrm{i}u, \mathrm{i}(u+v), 0),$$

solves

$$\eta_u = \tilde{U}\eta, \qquad \eta_v = \tilde{V}\eta, \tag{5.3}$$

where

$$\tilde{U} = \frac{1}{4\sqrt{2}}\begin{pmatrix} 0 & 0 & -4\mathrm{i} & 0\\ 0 & 4\sqrt{2}\mathrm{i} & 0 & 4\mathrm{i}\\ \mathrm{i}b & \mathrm{i}(a-1) & \sqrt{2}\mathrm{i}((a-1+b)+4) & \sqrt{2}\mathrm{i}(a-1-b)\\ \mathrm{i}(a-1) & -\mathrm{i}b & \sqrt{2}\mathrm{i}(a-1-b) & -\sqrt{2}\mathrm{i}(a-1+b)\end{pmatrix} \tag{5.4}$$

$$\tilde{V} = \frac{1}{4\sqrt{2}}\begin{pmatrix} 4\sqrt{2}\mathrm{i} & 0 & 0 & -4\mathrm{i}\\ 0 & 0 & -4\mathrm{i} & 0\\ \mathrm{i}(a-1) & -\mathrm{i}b & \sqrt{2}\mathrm{i}((a-1-b)+4) & -\sqrt{2}\mathrm{i}(a-1+b)\\ -\mathrm{i}b & -\mathrm{i}(a-1) & -\sqrt{2}\mathrm{i}(a-1+b) & -\sqrt{2}\mathrm{i}(a-1-b)\end{pmatrix}. \tag{5.5}$$

are constant. In particular, $\tilde{U}$ and $\tilde{V}$ are commuting matrices.

Proof. The systems of linear differential equations (5.2) and (5.3) are equivalent for

$$\tilde U = \mathrm{e}^{D}(D_u + U)\,\mathrm{e}^{-D} \quad \text{and} \quad \tilde V = \mathrm{e}^{D}(D_v + V)\,\mathrm{e}^{-D}.$$

One easily verifies

$$\mathrm{e}^{D} U\, \mathrm{e}^{-D} = \frac{1}{4\sqrt{2}}\begin{pmatrix} 0 & 0 & -4\mathrm{i} & 0 \\ 0 & 0 & 0 & 4\mathrm{i} \\ \mathrm{i}b & \mathrm{i}(a-1) & \sqrt{2}\mathrm{i}(a-1+b) & \sqrt{2}\mathrm{i}(a-1-b) \\ \mathrm{i}(a-1) & -\mathrm{i}b & \sqrt{2}\mathrm{i}(a-1-b) & -\sqrt{2}\mathrm{i}(a-1+b)\end{pmatrix}$$

so that $\tilde U$ is given by (5.4), and a similar computation gives $\tilde V$. Finally, since $\tilde U$ and $\tilde V$ are constant, the compatibility condition, $\eta_{uv} = \eta_{vu}$, shows that $\tilde U$ and $\tilde V$ are commuting. □

Since $\tilde U$ and $\tilde V$ are simultaneously diagonalizable, all solutions of (5.3) are of the form

$$\eta(u, v) = C\,\mathrm{e}^{D_1 u + D_2 v} c, \qquad c \in \mathbb{C}^4,$$

where C is a common basis of eigenvectors of $\tilde U$ and $\tilde V$, and D_1, D_2 are the corresponding diagonal matrices of eigenvalues.

Lemma 5.2.

(i) The spectra of $\tilde U$ and $\tilde V$ coincide, and

$$\operatorname{spec}(\tilde U) = \{\lambda_k | k \in \mathbb{Z}_4\}, \qquad \lambda_k := \lambda(i^k x).$$

Here we put $x := \mathrm{e}^{\frac{1}{4}\log(\mu)}$, where log *is the main branch of the logarithm, and*

$$\lambda(y) = \frac{(1+\mathrm{i})(y+1)(y+\mathrm{i})}{4y}$$

that is

$$\lambda_0 = \frac{(1+\mathrm{i})(x+1)(x+\mathrm{i})}{4x}, \qquad \lambda_1 = -\frac{(1-\mathrm{i})(x+1)(x-\mathrm{i})}{4x},$$
$$\lambda_2 = -\frac{(1+\mathrm{i})(x-1)(x-\mathrm{i})}{4x}, \qquad \lambda_3 = \frac{(1-\mathrm{i})(x-1)(x+\mathrm{i})}{4x}. \tag{5.6}$$

(ii) Let

$$w(y) = \begin{pmatrix} \frac{1}{\sqrt{2}}\xi(y) \\ \frac{1}{\sqrt{2}} \\ \mathrm{i}\xi(y)\lambda(y) \\ \mathrm{i}(\mathrm{i} - \lambda(y)) \end{pmatrix} \qquad \text{with} \quad \xi(y) := \mathrm{i}\frac{y-\mathrm{i}}{y+\mathrm{i}},$$

and define $w_k := w(\mathrm{i}^k x)$ and $\xi_k = \xi(\mathrm{i}^k x)$ where again $x = \mathrm{e}^{\frac{1}{4}\log\mu}$.

- *For $\mu \neq \pm 1$ the eigenvalues of $\tilde U$ (and $\tilde V$) are pairwise distinct. The eigenspaces of $\tilde U$ and $\tilde V$ are spanned by*

$$E_{\lambda_k}(\tilde U) = \operatorname{span}\{w_k\}.$$

- *For $\mu = 1$, the eigenvalues $\lambda_0 = \lambda_1 = \mathrm{i}$, $\lambda_2 = \lambda_3 = 0$ coincide, and the complex two-dimensional eigenspaces are given by*

$$E_{\lambda=i}(\tilde U) = \lim_{\mu\to 1} E_{\lambda_0}(\tilde U) \oplus E_{\lambda_1}(\tilde U),$$
$$E_{\lambda=0}(\tilde U) = \lim_{\mu\to 1} E_{\lambda_2}(\tilde U) \oplus E_{\lambda_3}(\tilde U).$$

- *For* $\mu = -1$*, the eigenvalues are* $\lambda_0 = \frac{1+\sqrt{2}}{2}\mathrm{i}$, $\lambda_2 = \frac{1-\sqrt{2}}{2}$ *and* $\lambda_1 = \lambda_3 = \frac{1}{2}\mathrm{i}$. *The eigenspaces are given by* $E_{\lambda_k}(\tilde{U}) = \operatorname{span}\{w_k\}$, $k = 0, 2$, *and*

$$E_{\lambda=\frac{1}{2}\mathrm{i}}(\tilde{U}) = \lim_{\mu\to -1} E_{\lambda_1}(\tilde{U}) \oplus E_{\lambda_3}(\tilde{U}),$$

where the latter is again complex two dimensional.

(iii) Let $\lambda_k \in \operatorname{spec}(\tilde{U})$ *be an eigenvalue of* $\tilde{U}$*, and define*

$$\epsilon_k := \xi_k \lambda_k = \lambda_{k+1}, \qquad k \in \mathbb{Z}_4.$$

Then ϵ_k *is an eigenvalue of* $\tilde{V}$*, and*

$$E_{\lambda_k}(\tilde{U}) = E_{\epsilon_k}(\tilde{V}).$$

We skip the computational proof [9] and remark that the group $\langle \zeta_4 \rangle = \langle \mathrm{i} \rangle$ acts on the spectrum by

$$\lambda(\sqrt[4]{\mu}) \mapsto \lambda(\mathrm{i}\sqrt[4]{\mu})$$

for some fourth root, $\sqrt[4]{\mu}$, of μ. For the subgroup $\langle \zeta_2 \rangle = \langle -1 \rangle$, the action can be described by

$$\lambda(-\sqrt[4]{\mu}) = \mathrm{i} - \lambda(\sqrt[4]{\mu}) \quad \text{resp.} \qquad \lambda_{k+2} = \mathrm{i} - \lambda_k, \quad k \in \mathbb{Z}_4.$$

Furthermore, we see that the eigenvalues are multi-valued functions in $\mu \in \mathbb{C}_*$ but are well defined on the 4 : 1-covering $\mathbb{C}^* \to \mathbb{C}^*$ given by $x \mapsto x^4 = \mu$. The group, $\langle \zeta_4 \rangle$, acts as deck transformations of this covering. We summarize

Proposition 5.3. *For each* $\mu \in \mathbb{C}^*$ *the fundamental parallel sections* $\hat{\varphi}_k := G\phi_k$, $k = 0, \ldots, 3$, *span the space of* d^μ*-parallel sections where*

$$\phi_k := \mathrm{e}^{-D} C\, \mathrm{e}^{D_1 u + D_2 v} e_k. \tag{5.7}$$

Here $e_k \in \mathbb{C}^4$ *is the* $(k+1)$*th standard basis vector,*

$$D = \operatorname{diag}(\mathrm{i}v, \mathrm{i}u, \mathrm{i}(u+v), 0),$$
$$D_1 = \operatorname{diag}(\lambda_0, \lambda_1, \lambda_2, \lambda_3),$$
$$D_2 = \operatorname{diag}(\epsilon_0, \epsilon_1, \epsilon_2, \epsilon_3),$$

and the columns of C *are the corresponding basis of eigenvectors of* $\tilde{U}$. *In particular, for* $\mu \neq 1$ *we get*

$$\phi_k = \begin{pmatrix} \frac{1}{\sqrt{2}}(\xi_k\, \mathrm{e}^{-\mathrm{i}v} + j\, \mathrm{e}^{-\mathrm{i}u}) \\ \mathrm{i}\epsilon_k\, \mathrm{e}^{-\mathrm{i}(u+v)} + j\mathrm{i}(\mathrm{i} - \lambda_k) \end{pmatrix} \mathrm{e}^{\lambda_k u + \epsilon_k v}$$

and for $\mu = 1$

$$\phi_0 = \begin{pmatrix} f \\ -1 \end{pmatrix}, \qquad \phi_1 = \frac{1}{\sqrt{2}} \begin{pmatrix} j \\ 0 \end{pmatrix}, \qquad \phi_2 = \phi_0 j, \qquad \phi_3 = \phi_1 \mathrm{i} j.$$

We now obtain all μ-Darboux transforms on the universal cover $\tilde{M} = \mathbb{C}$ of the Clifford torus.

Theorem 5.4. *Every* μ*-Darboux transform* $\hat{f} : \mathbb{C} \to S^4$ *of the Clifford torus,* $\mu \neq 1$*, is given by*

$$\hat{f}(u, v) = \frac{1}{\sqrt{2}}(g_1(u, v)\, \mathrm{e}^{\mathrm{i}u} + j g_2(u, v)\, \mathrm{e}^{\mathrm{i}v}),$$

where

$$g_1(u,v) = \frac{\sum_{k,l=0}^{3}(-(\mathrm{i}-\lambda_k)\overline{\lambda_l}(1+\xi_k\overline{\xi_l}))\,\mathrm{e}^{(\lambda_k+\overline{\lambda_l})u+(\epsilon_k+\overline{\epsilon_l})v}s_k\overline{s_l}}{\sum_{k,l=0}^{3}(\epsilon_k\overline{\epsilon_l}+(\mathrm{i}-\lambda_k)(\overline{\mathrm{i}-\lambda_l}))\,\mathrm{e}^{(\lambda_k+\overline{\lambda_l})u+(\epsilon_k+\overline{\epsilon_l})v}s_k\overline{s_l}}$$

$$g_2(u,v) = \frac{\sum_{k,l=0}^{3}(-(\mathrm{i}-\epsilon_k)\overline{\epsilon_l}(1+\xi_k\overline{\xi_l}))\,\mathrm{e}^{(\lambda_k+\overline{\lambda_l})u+(\epsilon_k+\overline{\epsilon_l})v}s_k\overline{s_l}}{\sum_{k,l=0}^{3}(\epsilon_k\overline{\epsilon_l}+(\mathrm{i}-\lambda_k)(\overline{\mathrm{i}-\lambda_l}))\,\mathrm{e}^{(\lambda_k+\overline{\lambda_l})u+(\epsilon_k+\overline{\epsilon_l})v}s_k\overline{s_l}}$$

with $s_k \in \mathbb{C}$.

Proof. Let $\mu \neq 1$ and $\phi = \sum_{k=0}^{3}\phi_k s_k$ be a parallel section of d^μ where $s_k \in \mathbb{C}$ and ϕ_k are the fundamental solutions (5.7). Then $\hat{f} = f + \alpha\beta^{-1}$ is the μ-Darboux transform given by $\phi = \binom{\alpha}{\beta}$, and the claim follows by a straightforward computation. □

Remark 5.5. In [2, theorem 2.5] it is shown that all immersed μ-Darboux transforms of a Willmore surface are again Willmore. In particular, the μ-Darboux transforms obtained above are Willmore surfaces in S^4.

So far, we considered the μ-Darboux transformation on the universal cover $\mathbb{C}$ of the 2-torus $T^2 = \mathbb{C}/\Gamma$. By lemma 2.4 we have to find parallel sections with multiplier to obtain closed Darboux transforms on the torus T^2. Since $\hat{\varphi} = G^{-1}\phi$, and G is defined (4.1) on $T^2 = \mathbb{C}/\Gamma$, it is enough to find solutions ϕ of (5.2) with multiplier.

Theorem 5.6. *Let* $f : \mathbb{C}/\Gamma \to S^3$ *be the Clifford torus.*

(i) A fundamental solution ϕ_k *is a parallel section of* d^μ *with multiplier, and the* μ*-Darboux transform given by* ϕ_k, $\mu \neq 1$, *is obtained by rotating and scaling* f. *For* $\mu = 1$, *all* μ*-Darboux transforms are constant.*

(ii) Let $\mu \neq 1$ *and* $\hat{f} : \mathbb{C}/\Gamma \to S^4$ *be a closed* μ*-Darboux transform of* f. *Then there exists a fundamental solution* $\hat{\varphi}_k = G\phi_k$ *with*

$$\hat{f} = \hat{\varphi}_k\mathbb{H}.$$

In particular, every non-constant μ*-Darboux transform* $\hat{f} : \mathbb{C}/\Gamma \to S^4$ *of* f *is the Clifford torus.*

Proof.

(i) If

$$\phi_k = \begin{pmatrix} \frac{1}{\sqrt{2}}(\xi_k\,\mathrm{e}^{-\mathrm{i}v} + j\,\mathrm{e}^{-\mathrm{i}u}) \\ \mathrm{i}\epsilon_k\,\mathrm{e}^{-\mathrm{i}(u+v)} + j\mathrm{i}(\mathrm{i}-\lambda_k) \end{pmatrix} \mathrm{e}^{\lambda_k u + \epsilon_k v}$$

is a fundamental solution, then the corresponding μ-Darboux transform is

$$\hat{f} = \frac{1}{\sqrt{2}}(r_1\,\mathrm{e}^{\mathrm{i}u} + r_2\,\mathrm{e}^{\mathrm{i}v}),$$

where

$$r_1 = \frac{|\epsilon_k|^2 + |\mathrm{i}-\lambda_k|^2 - \mathrm{i}\xi_k\overline{\epsilon_k} + \mathrm{i}(\mathrm{i}-\lambda_k)}{|\epsilon_k|^2 + |\mathrm{i}-\lambda_k|^2},$$

$$r_2 = \frac{|\epsilon_k|^2 + |\mathrm{i}-\lambda_k|^2 - \mathrm{i}\overline{\epsilon_k} - \overline{\xi_k}\mathrm{i}(\mathrm{i}-\lambda_k)}{|\epsilon_k|^2 + |\mathrm{i}-\lambda_k|^2}.$$

One easily verifies with $\epsilon_k = \xi_k\lambda_k$ and $\mathrm{i}-\lambda_k = -\xi_k(\mathrm{i}-\epsilon_k)$ that

$$\frac{r_1}{r_2} = -\frac{\xi_k}{\overline{\xi_k}} \in S^1,$$

so that $r_2 = r_1\,\mathrm{e}^{\mathrm{i}\theta}$ for a $\theta \in \mathbb{R}$ and $\hat{f}(u,v) = f(u, v+\theta)r_1$.

Proposition 5.3 implies that $\hat{\varphi}_k = G\phi_k$ is constant for $\mu = 1$, and thus an arbitrary solution, $\phi = \sum_k \phi_k s_k$, gives a constant Darboux transform $\hat{f} = G\phi\mathbb{H} = \text{const}$.

(ii) Let $\hat{f}$ be given by the section $\phi = G^{-1}\hat{\varphi}$ and suppose that ϕ is not a fundamental solution, i.e. $\phi = \sum_k \phi_k s_k$ and $s_k, s_l \neq 0$ for some $k \neq l$. The monodromy condition implies that

$$\phi(u + 2\pi, v) = \phi(u, v)h_1 \qquad \text{and} \qquad \phi(u, v + 2\pi) = \phi(u, v)h_2,$$

with $h_1, h_2 \in \mathbb{C}$. Since the fundamental solutions

$$\phi_k = \begin{pmatrix} \frac{1}{\sqrt{2}}(\xi_k \,\mathrm{e}^{-\mathrm{i}v} + j\,\mathrm{e}^{-\mathrm{i}u}) \\ \mathrm{i}\epsilon_k \,\mathrm{e}^{-\mathrm{i}(u+v)} + j\mathrm{i}(\mathrm{i} - \lambda_k) \end{pmatrix} \mathrm{e}^{\lambda_k u + \epsilon_k v}$$

are linearly independent over $\mathbb{C}$, it follows that

$$h_1 = \mathrm{e}^{2\pi\lambda_k} = \mathrm{e}^{2\pi\lambda_l} \qquad \text{and} \qquad h_2 = \mathrm{e}^{2\pi\epsilon_k} = \mathrm{e}^{2\pi\epsilon_l},$$

that is

$$\lambda_k - \lambda_l \in \mathrm{i}\mathbb{Z} \qquad \text{and} \qquad \epsilon_k - \epsilon_l = \lambda_{k+1} - \lambda_{l+1} \in \mathrm{i}\mathbb{Z}. \tag{5.8}$$

From (5.6) we see that

$$\lambda_0 - \lambda_1 = \frac{x^2 - 1}{2x}, \qquad \lambda_0 - \lambda_3 = \frac{\mathrm{i}(x^2 + 1)}{2x} \tag{5.9}$$

and the remaining differences $\lambda_k - \lambda_l$ can be computed by using $\Sigma_{k=0}^{3}(-1)^k\lambda_k = 0$. Then it is easy to show that (5.8) is satisfied only if $x \in \{\pm 1, \pm \mathrm{i}\}$ which contradicts $\mu = x^4 \neq 1$. □

6. New Willmore tori in S^4

As we have seen in theorem 5.6, the only μ-Darboux transforms of the Clifford torus on $\mathbb{C}/\Gamma$ are obtained by fundamental solutions $\hat{\varphi}_k$, and in this case the μ-Darboux transform is the reparametrized and scaled Clifford torus f. To obtain new examples, we consider an n^2-fold covering, $f : \mathbb{C}/\Gamma_n \to S^3, u + \mathrm{i}v \mapsto \frac{1}{\sqrt{2}}(\mathrm{e}^{\mathrm{i}u} + j\,\mathrm{e}^{\mathrm{i}v})$, of the Clifford torus with lattice $\Gamma_n = 2\pi n\mathbb{Z} + 2\pi n\mathrm{i}\mathbb{Z}$, and contemplate the μ-Darboux transforms of f.

Lemma 6.1. *Let $f : \mathbb{C}/\Gamma_n \to S^3$ be the n^2-fold covering of the Clifford torus. Then the following statements are equivalent:*

(i) For $\mu \in \mathbb{C}_*$ all μ-Darboux transforms are defined on $\mathbb{C}/\Gamma_n$.

(ii) $\mu = x^4$ with $x = \frac{p+\mathrm{i}q}{n} \in S^1$ and $(p, q) \in \mathbb{Z}^2 \setminus \{0\}$.

In this case, the multiplier $h : \Gamma_n \to \mathbb{C}^*$ is trivial, i.e. $h \equiv 1$.

Proof. Let $\phi = \sum_k \phi_k s_k$ be a parallel section of d^μ, $\mu = x^4$, with $s_k \neq 0$ for all k, where ϕ_k are the fundamental solutions (5.7). Then

$$\phi(u + 2\pi n, v) = \phi(u, v) \iff h = \mathrm{e}^{2\pi n\lambda_k} \qquad \text{for all} \quad k = 0, 1, 2, 3.$$

This implies that $n(\lambda_k - \lambda_l) \in \mathrm{i}\mathbb{Z}$ for all k, l, and as in the proof of theorem 5.6 it is enough to consider

$$n(\lambda_0 - \lambda_1) = \frac{n(x^2 - 1)}{2x} = \mathrm{i}p \qquad \text{and} \qquad n(\lambda_0 - \lambda_3) = \frac{\mathrm{i}n(x^2 + 1)}{2x} = \mathrm{i}q$$

for some $p, q \in \mathbb{Z}$. Using (5.9) we see that these equations can be satisfied if and only if $p^2 + q^2 = n^2$, that is $x = \frac{p+\mathrm{i}q}{n} \in S^1$. In this case

$$\lambda_k = \frac{\mathrm{i}(\pm p \pm q + n)}{2n} \qquad \text{for all } k.$$

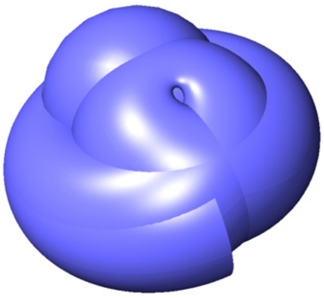

Figure 1. Willmore cylinder obtained by the μ-Darboux transformation.

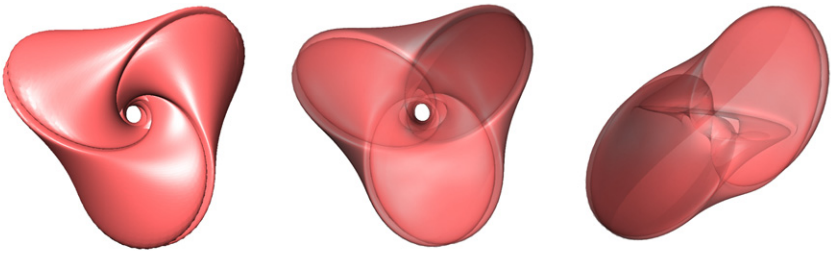

Figure 2. μ-Darboux transform with $(p, q, n) = (3, 4, 5)$.

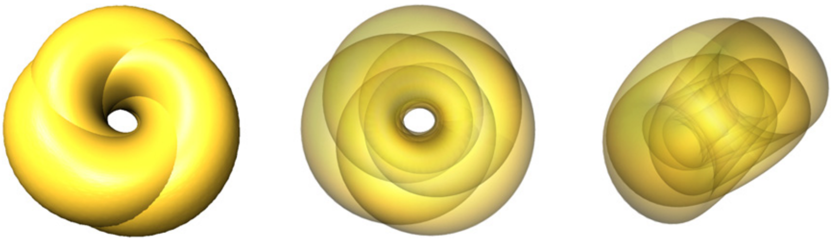

Figure 3. μ-Darboux transform with $(p, q, n) = (3, 4, 5)$.

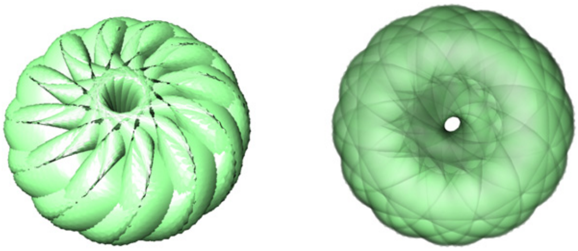

Figure 4. μ-Darboux transform with $(p, q, n) = (5, 12, 13)$.

For an arbitrary Pythagorean triple (p, q, n), it is known that $p \pm q$ and n are both odd, so that $\pm p \pm q + n$ is even and $h = \mathrm{e}^{2\pi n \lambda_k} = 1$. Since $\epsilon_k = \lambda_{k+1}$ we also see that the v-periods close. □

Since the Darboux transformation essentially preserves the geometric genus of the spectral curve and the Willmore energy [3] we have shown.

Theorem 6.2. *For all Pythagorian triple* (p, q, n), *there exists a* $\mathbb{CP}^3$ *family of* μ*-Darboux transforms* $\hat{f} : \mathbb{C}^2/\Gamma_n \to S^4$ *for* $\mu = \frac{p+iq}{n}$. *If* $\hat{f}$ *is immersed, then* $\hat{f}$ *is a Willmore torus with Willmore energy* $W(\hat{f}) = 2(\pi n)^2$. *Moreover, in this case* $\hat{f}$ *has spectral genus zero.*

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